JIAS

Journal of Intelligent Automation Systems
Published online: 07 January 2014

Phase Synchronization and Synchronization Frequency of Two Bi-Directionally Coupled Van der Pol

Hossein Gholizade-Narm [a,*], Morteza Khademi [b] and Asad Azemi [c]

Abstract: Phase synchronization phenomenon and synchronization frequency of the coupled Van der Pol oscillators were investigated through the use of describing the function method. The sufficient conditions of the phase synchronization were obtained analytically and the synchronization region was numerically plotted versus coupling intensity. The major advantage of the describing method over other methods is its ability to obtain the synchronization frequency. This is because the two coupled oscillators in the frequency of low, medium and high achieved were the Synchronous. This phenomenon was observed in the myocyte of the chicken heart. The simulation results confirm the theory discussion.

Keywords: Van der Pol Oscillator, Coupling, Describing Function, Phase Synchronization, Synchronization Frequency

a Faculty of Electrical Engineering, Shahrood University, Shahrood, Iran,, phone: +98(273)3300242(3213), Email: Gholizade@shahroodut.ac.ir , h_gholizade@yahoo.com
b Faculty of Engineering, Ferdowsi University of Mashhad, Iran
c College of Engineering, Penn State University, USA

1. Introduction

The history of synchronization goes back to the 17^{th} century, when the famous Dutch scientist Christian Huygens [1] reported his observation of synchronization of two pendulum clocks. The systematic study of this phenomenon, experimental as well as theoretical, was started by Edward Appleton [2] and Balthazar van der Pol [3]. They showed that the frequency of a triode generator can be entrained, or synchronized, by a weak external signal with slightly different frequency. These studies were of high practical importance because such generators became basic elements of radio communication systems [4]. More recently, two semiconductor lasers were found to exhibit synchronization of their light intensities [5]. Clusters of driven acoustic cavitation bubble simulations also showed synchronization effects [6].

The synchronization of two coupled oscillators has been studied using two methods: numerical and analytical. In the numerical method, at first, the systems phase is defined mathematically, then, the phase of each oscillator is numerically gained versus coupling intensity. If the phase difference of two oscillators were bound, two oscillators would be synchrony; otherwise, they are asynchrony [7-10]. The major drawback of this method was its dependence on a computer to solve the system equation numerically.

Among analytical methods, the perturbation methods, such as averaging and bi-variant expansion have been considered [11-13]. The main disadvantage of these methods was the inability to extract the synchronization frequency. However, the synchronization region could not be determined analytically if two oscillators were completely non-identical.

Another method, which has been used in recent years, is the describing function [14]. This method has the advantages of achieving to synchronization frequency. This method is used in this study.

Most studies conducted on synchronization so far concerned synchronization region versus coupling intensity, in which the synchronization frequency has not been considered. However, in practice, the synchronization frequency has been observed in chicken pacemaker cells [15]. In this paper, in addition to phase synchronization conditions, the frequency of oscillators after synchronization is discussed. Dehaan and Hirakow [15] reported that in the survey of a chicken heart, they observed that some pacemaker cells fire at a frequency lower than the slowest cell, some at a middle frequency, and the rest at a frequency higher than the highest frequency. This phenomenon is confirmed here analytically and demonstrated by several numerical examples.

The paper is organized as follow: Phase synchronization and its sufficient conditions are stated in section 2. System behavior and synchronization frequency are explained in section 3. Numerical simulations are given in section 4 and the conclusion is provided in section 5.

2. Phase Synchronization

The system under study is made up of two bi-directionally coupled Van der Pol oscillators. Assume the system equations as,

$$\ddot{x}(t)-\mu_1\left(1-x^2(t)\right)\dot{x}(t)+\omega_1^2x(t)=\alpha_1\left(y(t)-x(t)\right) \tag{1}$$
$$\ddot{y}(t)-\mu_2\left(1-y^2(t)\right)\dot{y}(t)+\omega_2^2y(t)=\alpha_2\left(x(t)-y(t)\right) \tag{2}$$

where μ_1 and μ_2 are damping coefficients, ω_1 and ω_2 are the natural frequencies and α_1 and α_2 are coupling intensities. By rearranging the equations, we have

$$\ddot{x}(t)-\mu_1\left(1-x^2(t)\right)\dot{x}(t)+\omega_1'^2x(t)=\alpha_1 y(t),\ \omega_1'^2=\omega_1^2+\alpha_1 \tag{3}$$
$$\ddot{y}(t)-\mu_2\left(1-y^2(t)\right)\dot{y}(t)+\omega_2'^2y(t)=\alpha_2 x(t),\ \omega_2'^2=\omega_2^2+\alpha_2 \tag{4}$$

The equations should be decomposed into two parts, linear and nonlinear for using describing function method.

$$\ddot{x}-\mu_1\dot{x}+\mu_1\dot{f}(x)+\omega_1'^2x=\alpha_1 y \tag{5}$$
$$\ddot{y}-\mu_2\dot{y}+\mu_2\dot{f}(y)+\omega_2'^2y=\alpha_2 x \tag{6}$$

where the nonlinear function $f(.)$ is,

$$f(\cdot)=\frac{1}{3}(\cdot)^3$$
$$\dot{f}=\frac{df}{dt}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} \tag{7}$$

Assume the synchronization frequency is ω''. Therefore, the signal which effects x has at least two main frequencies, ω_2' and ω''. Such a reason is for y, where the signal affecting y has frequencies, ω_1' and ω''. According to intrinsic and imposed frequency on x and y and considering only the first harmonic we have,

$$x(t)\approx x_1\sin\left(\omega_1't+\varphi_1'\right)+x_2\sin\left(\omega''t+\varphi_x\right)+x_3\sin\left(\omega_2't+\varphi_1''\right) \tag{8}$$

$$y(t)\approx y_1\sin\left(\omega_2't+\varphi_2'\right)+y_2\sin\left(\omega''t+\varphi_y\right)+y_3\sin\left(\omega_1't+\varphi_2''\right) \tag{9}$$

where $\varphi = [\varphi_1' \quad \varphi_x \quad \varphi_1'' \quad \varphi_2' \quad \varphi_y \quad \varphi_2'']$ are signals' phases. One can take one of the phases as the base phase and calculate the others versus the base.

The system block diagram is illustrated in Fig. 1.

The transfer function $G_x(p)$ is,

$$G_x(p) = \frac{\mu_1 p}{p^2 - \mu_1 p + \omega_1'^2} \tag{10}$$

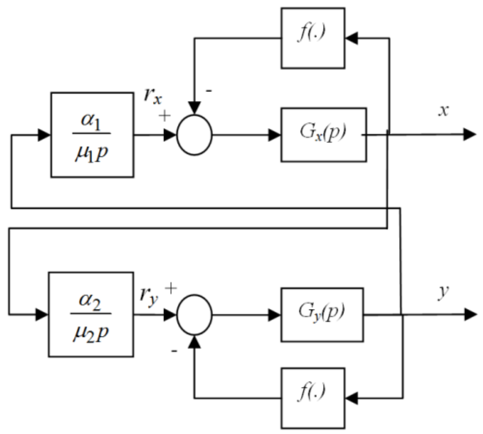

Fig. 1. The block diagram of two bi-directional coupled Van der Pol oscillators

Equation (10) shows that the transfer function is a band pass filter and attenuates high and low frequencies. Therefore, we can ignore frequencies such as $\omega_1' \pm \omega_2'$ and their harmonics as well as high frequencies.

According to (5) and (9) and considering the band pass transfer function $G(p)$,

$$r_x(t) \approx -\frac{\alpha_1 y_1}{\mu_1 \omega_2'} \cos(\omega_2' t + \varphi_2') - \frac{\alpha_1 y_2}{\mu_1 \omega''} \cos(\omega'' t + \varphi_y) - \frac{\alpha_1 y_3}{\mu_1 \omega_1'} \cos(\omega_1' t + \varphi_2'') \tag{11}$$

The signals x and y (eq. (8) and (9)) will be in phase synchrony if the amplitudes of various frequencies are much lower than synchronization frequency (ω'') amplitude. Let two oscillators work in synchrony. Consequently, the amplitude of y_1 and y_2 will be very little and we can properly approximate input signal as below,

$$r_x(t) \approx -\frac{\alpha_1 y_2}{\mu_1 \omega''} \cos(\omega'' t + \varphi_y) \tag{12}$$

For nonlinear function(s), we have [16],

$$\begin{aligned} f(x) &= \frac{1}{3}x^3 \approx N_{1x} x_1 \sin(\omega_1' t + \varphi_1') + \\ & N_{2x} x_2 \sin(\omega'' t + \varphi_x) + N_{3x} x_3 \sin(\omega_2' t + \varphi_1'') \\ N_{1x} &= \frac{1}{4}(x_1^2 + 2x_2^2 + 2x_3^2), N_{2x} = \frac{1}{4}(2x_1^2 + x_2^2 + 2x_3^2) \\ N_{3x} &= \frac{1}{4}(2x_1^2 + 2x_2^2 + x_3^2) \end{aligned} \tag{13}$$

Since $G(p)$ is a band-pass filter; therefore, only the fundamental harmonics are considered in $f(x)$.

The selection of nonlinear function as $f(x) = \frac{1}{3}x^3$ instead of $f(x, \dot{x}) = x^2 \dot{x}$ has three major advantages:

- The nonlinear part is static, therefore the describing function is independent from frequency,
- Function f(.) is odd and the bias consideration is not necessary for input-output consistency.
- The transfer function will be a band pass, Therefore the sub-harmonics (low frequencies such as $\omega_1' - \omega_2'$) can be ignored.

According to block diagram (Figure 1),

$$G_x^{-1}(j\omega)x + f(x) = r_x \tag{14}$$

where,

$$G_x^{-1}(j\omega) = \frac{j}{\mu_1}\left(\omega - \frac{\omega_1'^2}{\omega}\right) - 1 \tag{15}$$

By substituting (12) and (13) in (14) and rearranging them, we have [16],

$$\left(G_x^{-1}(j\omega_1') + N_{1x}\right)x_1 \approx 0 \tag{16}$$

$$\left(G_x^{-1}(j\omega'') + N_{2x}\right)x_2 \approx \frac{\alpha_1 y_2}{\mu_1 \omega''} \exp\left(-j\left(\frac{\pi}{2} + (\varphi_x - \varphi_y)\right)\right) \tag{17}$$

$$\left(G_x^{-1}(j\omega_2') + N_{3x}\right)x_3 \approx 0 \tag{18}$$

According to (13), (16) and (17),

$$\begin{aligned} & -1 + \frac{1}{4}(x_1^2 + 2x_2^2 + 2x_3^2) \approx 0 \\ & x_3 \approx 0 \end{aligned} \tag{19}$$

Equation (19) shows that x_1 will tend to zero if x_2 tends to $\sqrt{2}$. This phenomenon is called Quenching Phenomenon in oscillators [16]. Note that the above equation is approximately equal to but not exactly equal. Therefore, we can consider one of the sufficient conditions for synchronization as,

$$x_2 \geq \sqrt{2} \tag{20}$$

The same scenario is for y. Therefore, another sufficient condition for phase synchronization is,

$$y_2 \geq \sqrt{2} \tag{21}$$

Assume $\theta = \varphi_x - \varphi_y$ as the phase difference of two oscillators. By rearranging (17) and duplicating for y we have,

$$\left(\frac{j}{\mu_1}\left(\omega'' - \frac{\omega_1'^2}{\omega''}\right) - 1 + \frac{1}{4}x_2^2\right)x_2 \approx \frac{\alpha_1 y_2}{\mu_1 \omega''}\exp\left(-j\left(\frac{\pi}{2}+\theta\right)\right) \quad (22)$$

$$\left(\frac{j}{\mu_2}\left(\omega'' - \frac{\omega_2'^2}{\omega''}\right) - 1 + \frac{1}{4}y_2^2\right)y_2 \approx \frac{\alpha_2 x_2}{\mu_2 \omega''}\exp\left(-j\left(\frac{\pi}{2}-\theta\right)\right) \quad (23)$$

The set of equations (22) and (23) consist of four equations and four unknown parameters (ω'', θ, x_2, y_2). Considering the real and imaginary parts of equations, suppose two oscillators are in phase synchrony. The real part of (22) and (23) is,

$$\begin{aligned} &\left(-1+\frac{1}{4}x_2^2\right)x_2 = -\frac{\alpha_1 y_2}{\mu_1\omega''}\sin(\theta) \\ &\left(-1+\frac{1}{4}y_2^2\right)y_2 = \frac{\alpha_2 x_2}{\mu_2\omega''}\sin(\theta) \end{aligned} \quad (24)$$

If $0<\theta<\pi$, which means that the phase of x is leader than phase of y, since all parameters are positive, we will have,

$$\begin{aligned} &-1+\frac{1}{4}x_2^2 < 0 \Rightarrow x_2 < 2 \\ &-1+\frac{1}{4}y_2^2 < 0 \Rightarrow y_2 > 2 \end{aligned} \quad (25)$$

According to (25) and constraints (20) and (21), we have,

$$\sqrt{2} \le x_2 < 2 \quad (26)$$

$$y_2 > 2 \quad (27)$$

This situation occurs when $\omega_1 > \omega_2$.

It is clear that if $\pi < \theta < 2\pi$, the position of parameters x_2 and y_2 is exchanged in the aforementioned equations. Example (1) simulates this case. Since the amplitude of x and y can be approximated properly with x_2 and y_2, when they are in phase synchrony, x_2 and y_2 are called as x and y amplitudes in all simulations.

As mentioned earlier, the conditions (26) and (27) are sufficient conditions for phase synchronization. Since these equations are based on amplitudes and not system parameters, we try to find sufficient conditions for phase synchronization based on system parameters. In order to achieve this goal, we have to find amplitudes based on system's parameters. Using (22) and (23), we have,

$$-\left(-1+\frac{1}{4}x_2^2\right)x_2\frac{\mu_1\omega''}{\alpha_1 y_2} = \left(-1+\frac{1}{4}y_2^2\right)y_2\frac{\mu_2\omega''}{\alpha_2 x_2} = \sin(\theta) \quad (28)$$

$$\left(\omega'' - \frac{\omega_1'^2}{\omega''}\right)\frac{x_2\omega''}{\alpha_1 y_2} = \left(\omega'' - \frac{\omega_2'^2}{\omega''}\right)\frac{y_2\omega''}{\alpha_2 x_2} = -\cos(\theta) \quad (29)$$

Equation (28) shows that,

$$\left(-1+\frac{1}{4}x_2^2\right)\alpha_2\mu_1 x_2^2 + \left(-1+\frac{1}{4}y_2^2\right)\alpha_1\mu_2 y_2^2 = 0 \quad (30)$$

From (29), x_2 is a function of y_2 as,

$$\begin{aligned} &\frac{1}{4}\alpha_2\mu_1 x_2^4 - \alpha_2\mu_1 x_2^2 + a = 0 \Rightarrow \\ &x_2^2 = \frac{\alpha_2\mu_1 \pm \sqrt{(\alpha_2\mu_1)^2 - a\alpha_2\mu_1}}{\frac{1}{2}\alpha_2\mu_1} \\ &\quad = 2 \pm \frac{2}{\alpha_2\mu_1}\sqrt{(\alpha_2\mu_1)^2 - a\alpha_2\mu_1} \\ &a = \left(-1+\frac{1}{4}y_2^2\right)\alpha_1\mu_2 y_2^2 \end{aligned} \quad (31)$$

For $0<\theta<\pi$ case, y_2 is greater than 2; therefore, a is positive. $x_2 > \sqrt{2}$ from (26) is

$$x_2^2 = 2 + \frac{2}{\alpha_2\mu_1}\sqrt{(\alpha_2\mu_1)^2 - a\alpha_2\mu_1} \cong A(y_2) \quad (32)$$

Condition (26) states that,

$$x_2 < 2 \Rightarrow \frac{1}{\alpha_2\mu_1}\sqrt{(\alpha_2\mu_1)^2 - a\alpha_2\mu_1} < 1 \quad (33)$$

x_2 is a real parameter, therefore we must have,

$$(\alpha_2\mu_1)^2 - a\alpha_2\mu_1 \ge 0 \Rightarrow a \le \alpha_2\mu_1 \quad (34)$$

Equations (31) and (34) result,

$$\begin{aligned} &\frac{1}{4}\alpha_1\mu_2 y_2^4 - \alpha_1\mu_2 y_2^2 - \alpha_2\mu_1 \le 0 \Rightarrow \\ &2 - \frac{2}{\alpha_1\mu_2}\sqrt{(\alpha_1\mu_2)^2 + \alpha_1\alpha_2\mu_1\mu_2} \le y_2^2 \\ &y_2^2 \le 2 + \frac{2}{\alpha_1\mu_2}\sqrt{(\alpha_1\mu_2)^2 + \alpha_1\alpha_2\mu_1\mu_2} \end{aligned} \quad (35)$$

According to (27), $y_2 > 2$ therefore,

$$4 \le y_2^2 \le 2 + \frac{2}{\alpha_1\mu_2}\sqrt{(\alpha_1\mu_2)^2 + \alpha_1\alpha_2\mu_1\mu_2} \quad (36)$$

The equation (36) is simplified if $\alpha_1\mu_2 = \alpha_2\mu_1$,

$$4 \le y_2^2 \le 2 + 2\sqrt{2} \tag{37}$$

The x_2 parameter was obtained based on y_2. Using (29), the synchronization frequency is obtained based on y_2,

$$\omega''^2 = \frac{\omega_1'^2 \alpha_2 A(y_2) - \omega_2'^2 \alpha_1 y_2^2}{\alpha_2 A(y_2) - \alpha_1 y_2^2} \cong B(y_2) \tag{38}$$

Adding the square of (28) and (29),

$$A(y_2)\left(\left(B(y_2) - \omega_1'^2\right)^2 + \left(-1 + \frac{1}{4}A(y_2)\right)\mu_1^2 B(y_2)\right) = \alpha_1^2 y_2^2 \tag{39}$$

Equation (39) shows a set of one equation-one unknown parameter. By solving this nonlinear set, y_2 is obtained based on system's parameters. By substituting y_2 in (32) and (38), x_2 and ω'' are calculated. Substituting x_2 and y_2 in (33) and (36), the sufficient conditions for phase synchronization are gained based on system's parameters. It is important to note that conditions (33) and (36) are obtained based on the assumption that $0 < \theta < \pi$. If $\pi < \theta < 2\pi$, the similar scenario must be repeated.

Unfortunately, equation (39) is a complicated nonlinear parametric equation; therefore, analytical solution is impossible, but it can be solved numerically for numerical examples. Example (2) illustrates the synchronization region based on coupling intensities (α_2, α_1).

3. Synchronization Frequency

In this section, the frequency of synchronization mode versus variation of phase difference, caused by variation of coupling intensity, is discussed. The imaginary part of (22) and (23) are studied for this purpose.

$$x_2\left(\omega'' - \frac{\omega_1'^2}{\omega''}\right) = -\frac{\alpha_1 y_2}{\omega''}\cos(\theta) \tag{40}$$

$$y_2\left(\omega'' - \frac{\omega_2'^2}{\omega''}\right) = -\frac{\alpha_2 x_2}{\omega''}\cos(\theta) \tag{41}$$

If $0 < \theta \le \frac{\pi}{2}$ which occurs when $\omega_1 \ge \omega_2$ and coupling intensity is big, or if $-\frac{\pi}{2} < \theta \le 0$ which means $\omega_1 \le \omega_2$ and coupling intensity is great, we have,

$$\omega'' \le \min(\omega_1', \omega_2') \tag{42}$$

Equation (42) states that two oscillators are synchronized at low frequency. If $\omega_2' \le \omega_1$ ($\omega_1' \le \omega_2$), the synchronization frequency is located between two natural frequencies $\omega_2 \le \omega'' \le \omega_1$ ($\omega_1 \le \omega'' \le \omega_2$) or becomes lower than two natural frequencies. The simulation result shows that it is generally located between two natural frequencies. Example 2 in section 4 simulates the aforementioned discussion.

If $\frac{\pi}{2} < \theta \le \pi$ which occurs when $\omega_1 \ge \omega_2$ and coupling intensity is small, or if $\pi < \theta \le \frac{3\pi}{2}$ which means $\omega_1 \le \omega_2$ and coupling intensity is small, we have,

$$\omega'' \ge \max(\omega_1', \omega_2') \tag{43}$$

Equation (43) states that two oscillators are synchronized at a frequency higher than two natural frequencies. Example 3 in section 4 simulates synchronization at high frequency.

The equations (42) and (43) justify several practical phenomena,

- In their observation of the myocyte of the chicken heart Dehaan and Hirakow [11] found that some cells fire at a frequency higher than the highest cells frequency, a number of them fire at middle frequency and the rest, at a frequency lower than the lowest cell frequency. This phenomenon occurs in respect to variation of coupling intensity.
- The Sinu-Atrial Node is composed from thousands of pacemaker cells, which are coupled bi-directionally. If the coupling intensity decreases, the rate of impulse generation will increase and the sinus tachycardia will arise, and if the coupling intensity increases very much, the sinus bradycardia will take place.

The phase difference $\theta = \pm 90^\circ$ is the critical point. At this point the frequency switching occurs. From (42) and (43) we can obtain a line based on (α_1, α_2) which allows the necessary condition to have $\theta = \pm 90^\circ$.

$$\sqrt{\omega_1^2 + \alpha_1^*} = \sqrt{\omega_2^2 + \alpha_2^*} \Rightarrow \alpha_2^* = \alpha_1^* + \omega_1^2 - \omega_2^2 \tag{44}$$

Equation (44) demonstrates that if $\omega_1 \ge \omega_2$ then must $\alpha_1 \le \alpha_2$ and vise versa. The (44) is a necessary condition, but not sufficient. Because there are an infinite number of small α_1 and α_2 which satisfy (44), but cannot synchronize two oscillators and there are numerous large α_1 and α_2 which satisfy (44), the absolute value of phase difference is less than 90°. Equation (44) is a sufficient condition only for uni-directional coupling in which one of coupling intensities has to be zero. If the condition (44) is held, the switching can happen softly because there is no gap between $\min(\omega_1', \omega_2')$ and $\max(\omega_1', \omega_2')$. This case is simulated in example 4.

4. Simulation Results

In this section, some examples are illustrated for clarity.

Example 1. Assume two coupled oscillators' parameters as,

$$\omega_1 = 1.6, \mu_1 = 2, \alpha_1 = 2, \omega_2 = 1, \mu_2 = 1, \alpha_2 = 2$$

The timeline of *x* and *y* are illustrated in Figure 2.

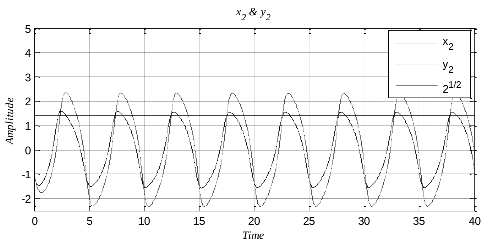

Fig. 2. Phase Synchronization of two oscillators

Figure 2 demonstrates that the amplitude of *y* is greater than 2 and the amplitude of *x* is between $\sqrt{2}$ and 2 (Eqns. (26) and (27)).

Example 2. Consider the system parameters, for Ex. 1,

$$\omega_1' = \sqrt{\omega_1^2 + \alpha_1} = 2.1354, \omega_2' = \sqrt{\omega_2^2 + \alpha_2} = 1.7321$$

The synchronization frequency can be obtained from Figure 2 as $\omega'' = 1.2320 < \min(\omega_1', \omega_2') = 1.7321$ that states the synchronization frequency is between two natural frequencies ($\omega_2 < \omega'' < \omega_1$). The phase difference is illustrated in Figure 3. It is obvious that the phase difference is less than $\frac{\pi}{2} = 1.5708$ (Eq. (42)).

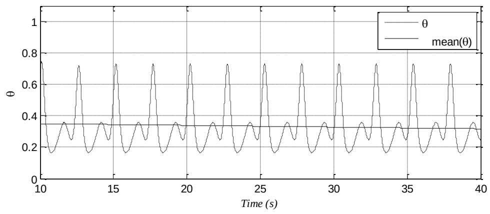

Fig. 3. The phase difference of two oscillators

The oscillations in the phase difference are caused by sub and super harmonics and the problems of method for phase extraction.

Example 3. Assume coupling intensities as $\alpha_1 = \alpha_2 = 1.3$ and the rest is previously mentioned. We have,

$$\omega_1' = \sqrt{\omega_1^2 + \alpha_1} = 1.9647, \omega_2' = \sqrt{\omega_2^2 + \alpha_2} = 1.5166$$

Figure 4 illustrates synchronization at high frequency. The synchronization frequency is $\omega'' = 2.001 > \max(\omega_1', \omega_2') = 1.9647$ which is higher than two natural frequencies ($\omega_2 < \omega_1 < \omega''$) (Eq. (43)).

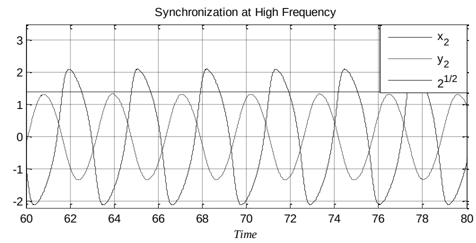

Fig. 4. Phase synchronization at high frequency

The phase difference is demonstrated in Figure 5. It is clear that $\frac{\pi}{2} = 1.5708 < \theta < \frac{3\pi}{2} = 4.7124$.

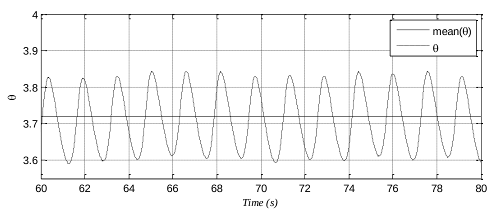

Fig. 5. Phase difference at high frequency

Example 4. Assume two oscillators parameters as $\omega_1 = 1.6, \mu_1 = 2, \alpha_1 = 0, \omega_2 = 1, \mu_2 = 1$. According to (44), $\alpha^* = 1.56$. Figure (6) illustrates phase difference for $\alpha < \alpha_2^*$, $\alpha = \alpha_2^*$ and $\alpha > \alpha_2^*$.

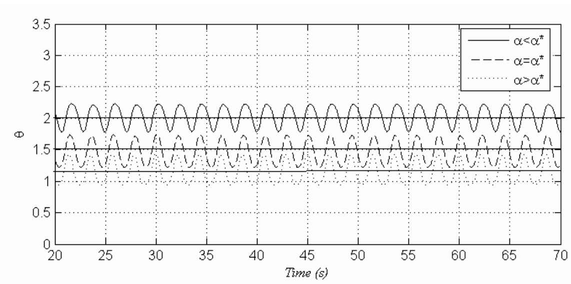

Fig. 6. Phase difference for uni-directional coupled oscillators

Phase differences according to Figure 6 are $\theta_1 = 2.022, \theta_2 = \theta^* = 1.503 \approx \frac{\pi}{2}, \theta_3 = 1.166$ respectively. It is clear that the phase difference decreases when the coupling intensity increases (Eq. (44)).

Example 5. Assume the parameters of oscillators as $\omega_1 = 1.6, \mu_1 = 2, \omega_2 = 1, \mu_2 = 1$, The synchronization area is illustrated is Figure 7, which is obtained from a numerical solution.

$\omega_1 = 1.6, \mu_1 = 2, \omega_2 = 1, \mu_2 = 1$

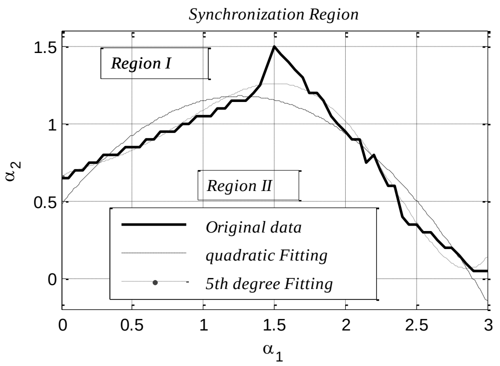

Fig. 7. Synchronization region for two bi-directional coupled oscillators

Figure 7 shows that the plane is decomposed into two areas of I and II. If coupling intensities occur in I, the oscillators will be synchronized. Otherwise, they will oscillate in asynchrony mode. The other two curves are quadratic and 5th degree fitting for the isolator points. The equations of two curves are,

$$QuadraticPolynomial:$$

$$\alpha_2 = -0.4394\alpha_1^2 + 1.1075\alpha_1 + 0.48376 = p_1(\alpha_1)$$

$$5th\ Degree\ Polynomial:$$

$$\alpha_2 = 0.13\alpha_1^5 - 0.74\alpha_1^4 + 1.2\alpha_1^3 - 0.51\alpha_1^2 + 0.38\alpha_1 + 0.67 = p_2(\alpha_1)$$

According to fitting curves, the synchronization condition is,

$$\alpha_2 \geq p_1(\alpha_1)\ Or\ \alpha_2 \geq p_2(\alpha_1)$$

As a result, the two oscillators can be phase synchronized if the coupling intensities hold the condition above.

5. Conclusion

When two oscillators with different natural frequencies are coupled, according to coupling intensity, they can be synchronized or oscillate asynchrony. If the synchronization happens, it is expected that the synchronization frequency locates between two natural frequencies of oscillators. In other words, the rate of rapid oscillator decreases and the rate of slow oscillator increases until the oscillation rate of two oscillators becomes the same. In this paper, it has been made known that this case generally happens when the coupling intensities have an average value. However, when the coupling intensities decrease, the synchronization frequency switches to a high frequency, which is higher than the two oscillators frequency. This case has been observed in the myocyte cells of the chicken heart. This result can also justify tachyarrhythmia, when the SA node pacemaker cells discharge rapidly. On the other hand, the synchronization frequency takes a low value when the coupling intensities increase. This result may also be an explanation for the bradicardia in the heart when the heart rate is slow. The describing function method used in this paper, and has the ability to extract the synchronization frequency. To employ this method, the system was properly decomposed into two subsystems as linear and nonlinear parts in a way that the linear part acts as a band-pass filter, which attenuates low and high frequencies and especially high order harmonics. This matter simplifies the analysis. Hence, the nonlinear part becomes a static odd function consequently; the describing function will be independent of frequency.

References

[1] C. Huygens, Horoloquium Oscilarium, Paris, 1673.

[2] E. V. Appleton: The automatic synchronization of triode oscillator, Proc. Cambridge Phil. Soc. (Math. and Phys. Sci.) 21, 231–248 (1922).

[3] B. van der Pol: Forced oscillators in a circuit with non-linear resistance. (Reception with reactive triode), Phil. Mag. 3, 64–80 (1927).

[4] I. Wedekind and U. Parlitz: Synchronization and antisynchronization of chaotic power drop-outs, Phys. Rev. E 66, 026218,1–4 (2002).

[5] J. T. Tervo, R. Mettin and W. Lauterborn, Bubble cluster dynamics in acustic cavitation: Acta Acustica united with Acustica 92(1), 178–180 (2006) .

[6] A. Pikovsky, M. Rosenblum, and J. Kurths: Phase synchronization in regular and chaotic systems, International Journal of Bifurcation and Chaos, 10(10), 2291–2305(2000).

[7] M. G. Rosenblum, A. S. Pikovski, and J. Kurths: Phase synchronization of chaotic oscillators, Phys. Rev. Lett. 76, 1804–1807 (1996).

[8] A. M. D. Santos, S. R. Lopes, R. L. Viana: Rhythm synchronization and chaotic modulation of coupled Van der Pol oscillators in a model for the heartbeat, Physica A, 338, 335–355 (2004).

[9] H. Fuhisaka, T. Yamada, G. Kinoshita, and T. Kono: Chaotic phase synchronization and phase diffusion, Physica D, 205, 41–47 (2005)

[10] M.G. Rosenblum and J. Kurths: Analysing Synchronization Phenomena from Bivariate Data by Means of the Hilbert Transform, in: Nonlinear Analysis of Physiological Data, Springer Series for Synergetics, (Springer, Berlin), 91–99 (1998)

[11] R. H. Rand, Lecture notes in nonlinear vibrations (version 45). Ithaca, NY: The Internet-First University Press, http://dspace.library.cornell.edu/handle/1813/79 , 2004

[12] X. Li, J.C. Ji , C. H. Hansen: Dynamics of two delay coupled van der Pol oscillators, Mech. Res. Comm., 33, 614–627 (2006).

[13] S. Wirkus, R. Rand: The Dynamics of Two Coupled van der Pol Oscillators with Delay Coupling, Nonlinear Dynamics, 30(3), 205–221 (2002).

[14] H. Gholizade-Narm, A. Azemi, and M. Khademi: Phase synchronization and synchronization frequency of two-coupled van der Pol oscillators with delayed coupling, Chin. Phys. B 22(7), 2013.

[15] R. L. Dehaan, and R. Hirakow: Synchronization of pulsation rates in isolated cardiac myocytes, Exp. Cell. Res., 70, 214–220 (1972).

[16] P. A. Cook: Nonlinear dynamical systems, Prentice Hall International, 1986.